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Questioning the Exclusivity of Classical Logic and Set-Theoretic Assumptions in Analysis of Classroom Argumentation and Proof

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The paper highlights the necessity to question the exclusivity of classical logic, or of approaches that are reducible to it, in the analysis of classroom proof and argumentation processes, as well as the role of the set-theoretic language as intrinsically linked to classical logic. Two examples drawn from mathematics classroom are analysed, recurring to the Ancient Indian empiricist Nyaya logic and to Peirce's non-standard quantification, associating the last to a "free logic", not axiomatizable within an axiomatic system where the specification axiom applies.

Keywords: Logic, Non-standard quantification, Nyaya, Free logic, Set-theoretic language.

Introduction.

The kind of logic students spontaneously resort to when they conjecture, argue or proof in mathematics classroom is often difficult to capture with the formal instruments of propositional logic (Barrier et al., 2009). Some scholars propose natural deduction for First Order Logic (FOL) as useful to reduce the distance between non deductive argumentation schemes and mathematical proof because of the possibility it offers to work on objects rather than on properties (Durand-Guerrier, 2005). To capture reasoning in Mathematics Education (ME) also Hintikka's dialogical logic, in reference to game theoretic semantic, is studied (e.g., Arzarello & Soldano, 2019; Blossier et al., 2009). What is common to these approaches is that they all are classical[1] or are reducible to the classical one.[2] Now, as Lindström's theorems shows, classical logic is intrinsically connected to set-theoretic language (Zalamea, 2021). In classical FOL the variables the quantifiers refer to, range on sets that represent the domains of the predicates. One of the fundamental axioms of set theory[3] is the axiom of specification: given a set A and a formula φ(x), there exists a subset B={a ∈ A: φ(a)}. This axiom is based on Frege's symmetry principle according to which one obtains "an equivalence […] (locally, within the restricted universe A) between φ(a) (intensionality) and a ∈ B (extensionality)" (Zalamea, 2009/2012, p. 324). If this axiom fails, both the law of the excluded middle (thus classical logic) and the standard use of quantifiers fail, because it is not guaranteed that a property univocally determines a set. From the other hand, the domain of reference of the statements during a learning process evolves over time and to grasp this evolution, sets should become "variable" (Lawvere & Rosebgough, 2003). Such sets can be captured by topoi in intuitionistic logic, considering an

[1] Classical logic is the logic where the law of the excluded middle ($A \vee \neg A$) and the law of non-contradiction ($\neg(A \wedge \neg A)$ hold, while non-classical logics are logics where at least one of these two characteristic properties does not hold. Examples of non-classical logics are the paraconsistent logic (the principle of non-contradiction holds only locally but not globally) and the intuitionistic logic (the law of the excluded middle does not hold and consequently also the double negation does not mean in general an assertion: $\neg\neg A \nvdash A$).

[2] The concept of truth in the game-theoretic semantic is different from the classical one (it is based on the logical existence of a choice function that guarantees the existence of a winning strategy for one of the players, called the verifier), but it can be shown that this truth concept is equivalent to the Tarskian one (Arzarello & Soldano, 2019) and thus to the truth conception in classical bivalent logic that follows the Aristotelian tradition.

[3] I refer to the Zermelo-Fraenkel axiomatic system with the axiom of choice (ZFC), as it is the standard axiomatic set–theoretic system within which mathematics usually is developed.

evolution over time, but not by classical sets. Classical sets, and thus classical logic, could be considered as special cases where time collapses into a moment and sets become fix.

Summing up, since classical FOL is exactly tailored to capture classical set theory, restriction to set-theoretical language may not allow different kinds of rationalities, that need to give up some of the constrains of set-theory, to be recognised, and thus it prevents also the investigation of possible shifts between "non-standard" and classical rationalities. Indeed, such "epistemic" rationalities require to consider indeterminacy about the properties that hold or do not hold for an object. I argue, recurring to examples, that non-classical approaches to logic and quantification, which don't require set-theoretic assumptions, could be able to put into evidence these aspects in the analysis of reasoning in mathematics classroom. In this way, (at least) novices' reasoning in ME, even if it does not match to classical logic, could be recognized as *knowledge* within a suitable rationality frame (Boero, 2017), rather than as a lack of rationality.

Theoretical framework.

Nyaya and empiric rationality. In the Western mathematical tradition, the Aristotelian syllogism represents the basis of logical reasoning and for mathematical proofs only the deductive syllogistic inferences are accepted. On the other hand, D'Amore (2005) shows that when dealing with proof, novice students might spontaneously resort to a type of logic very different from the Aristotelian one– the Indian Nyaya logic, a pragmatic and empiricist logic, linked to perception. In the Nyaya induction and deduction are closely interconnected within its "syllogism". Furthermore, the use of examples is not only permitted but is expected by the argumentative model itself and the "formal" and "material" aspects are closely intertwined in it (Sharma, 1962, p. 186), for the inferential model itself is conceived as a proof process of truth. According to D'Amore (2005), the Indian Nyaya philosophical school (1st century BC) recognizes a pre-eminent importance to four *means of knowledge*: testimony, analogy, perception and inference. The inference is what can be considered the Nyaya "syllogism" and has the following structure: (1) the Assertion (what one wants to prove); (2) the Reason; (3) the Thesis (a general proposition followed by an example); (4) the Application; (5) the Conclusion. Finally, one of the fallacies of the "right reasoning" in Nyaya is reasoning on non-existent objects.

Peirce's non-standard quantification. In ME also non-standard quantification, that cannot be framed within classical FOL, is epistemologically accounted (Blossier et al, 2009), with the aim to explain difficulties in managing quantification in classical sense at tertiary level or in the shift from secondary to tertiary level. These authors show that expert students (at tertiary level) spontaneously use different kinds of quantification that often involves temporal aspects and a kind of variation of the variables that often do not fit with the $\exists\forall$-variation as it is known after the introduction of the axiom of choice. They mention within the non-standard approaches to quantification Bolzano's (link between constant and variable quantities) and Cauchy's (link between variable quantity and fixed limit) ones, but they also account for the Peircean one, putting into evidence that it does not rest on logical distinctions but is "inner to the individuum" (Blossier et al., p. 84). I will deepen this last non-standard approach.

According to Peirce, quantification can be general, vague, or precise. Peirce calls generality, vagueness, and determination "the three affections of terms, [which] form a group dividing a category of what Kant calls 'functions of judgment'" (Peirce, CP, 5.450)[4]. *Generality* means absence of distinction of individuals rather than validity for every individual, as it is the case for the classical

[4] Peirce's Collected Papers (CP) are quoted in the usual way: (Peirce, CP, volume number.paragrph number).

universal quantifier that quantifies over sets of individuals; it can be expressed by words like *any*, *whatever*, etc. *Vagueness* means a certain type of existence that does not break the absence of distinction of individuals, but states that there are suitable generic individuals that satisfy a certain property; it can be expressed by words like *some*, *certain*, etc. It is similar to the classical existential quantifier but while the genericity of the latter rests on the proof of independence from the choice of a specific individual, the former rests on the *knowledge of the possibility* to choose individuals that remain indistinct, without a real actualization. *Precision* means effective actualization of possibility; the precise individual represents a rupture of the relationality that distinguishes the vagueness. As Hintikka's logic also the Peircean one is a dialogic logic with a game-theoretic semantic (Pietarinen, 2019), but Peirce's logic is epistemic in a different manner as Hintikka's one. Indeed, as Zalamea (2021) shows, Peirce's logic can be captured by sheaf-logic and sheaf-logic is intuitionistic. Thus, quantification in Peirce's logic does not require the axiom of specification and the symmetry of Frege's abstraction principle fails in general. Furthermore, according to Hintikka (2001) intuitionistic logic is truly epistemic because the crucial notion in it: "is not *knowing that*, but *knowing what* (*which, who, where, …*), in brief, *knowing* + an indirect question, that is, knowledge of objects rather than knowledge of truth" (p. 10) and this *knowing-what*-logic "cannot be analysed in terms of *knowing that* plus the apparatus of received first order logic" (p. 11).

The Nyaya logic is an example of an empiricist logic where reasoning applies on single objects, considered as "existent" by the subject; Peirce's logic with its non-standard quantification can be considered as an example of *free logic,* where the domain the quantifiers range over is not necessarily a closed set but "the class of existing things" (Nolt, 2021). In this sense, these two approaches are compatible and can be combined, at least at the basic level considered in this context.

Methodology of research.

A hermeneutical approach to the text analysis (Palmer, 1969; Bagni, 2009) is adopted. In this approach, the procedure consists in a dialectical back and forth between the meaning of the single parts of a text (oral, written etc.) and its global sense, in a meaning-increasing dialectical interpretation. The begin of the interpretation is always based on the interpreter's presuppositions about the original context of the analysed text (cultural, historical, etc.). The concept of *personal space* (Brown, 1996) of the protagonists (students and teacher) is used to frame the researcher's presuppositions in entering the analysis of the classroom excerpts and in searching for a global meaning, going from the part (examples) to the whole (discussions and conclusions) and vice versa. According to Brown, the *personal space* is the (virtual) space where "an individual sees him or her self acting" (p. 120); it is made by all the aspects, interests, constraints and means that inform the subject's acting in a context and is a source for meaning because "the individual acts in the world he or she imagines to exist" (p. 121). Here it mirrors the students' and teacher's background, inferred by the cultural context they are merged in while making mathematical statements or orchestrating mathematical classroom activities.

Data analysis and discussion.

Example 1: Empiric rationality, Nyaya, and non-standard quantification.

In this section an argumentative text produced by a 15-year-old high school student is discussed. S/he should answer the question: Is it true that *Each number that ends with the digit 1 is a prime number (that means without divisors different from 1 and the number itself) or it is divisible by 3*? The teacher's approach is Aristotelian and her and the student's personal spaces are inferred from information provided by the researcher that collected the data[5]. They are framed by the personal backgrounds (professional and formative), as well as by the classroom context.

In the analysis (Figure 1) classical Aristotelian and Nyaya-lenses are adopted: student's words are marked in **black bold**; the classification based on the Nyaya-scheme in green; the interpretation based on the Nyaya rationality frame in orange and the one within the Aristotelian frame in blue[6].

Assertion. **It is true,**
Statement whose truth is intended to be proven, as well as the expression of an intuition / Expression of an intuition.

Reason. **in fact, if we consider 11, 21, 31, 41, 51, 61, 71, etc. we see that they are all prime numbers or numbers divisible by 3.**
Reason why the statement is true. / Examples.

-------- **On the other hand, it is possible to reason also in general terms.**
Introduction of the general statement / Introduction of the generalization.

Thesis 1. **If a number is prime and greater than 2, it ends with an odd digit**
Shows the existence of the object he is talking about.

Thesis 2. **As for 3, among its multiples there are 21, 51, 81, 111, 141, etc.,**
Shows the existence of the object by showing generic examples.

Reverses thesis and hypothesis.

Application 1. **(and 1 is just an odd digit).**

Application 2. **all numbers ending on the right with the number 1.**

It is shown that the conditions of the Reason are fulfilled. / Wrong reasoning due to the inversion between thesis and hypothesis; tries to prove the inverse statement.

Conclusion. **The thesis has been proven.**
It has been "proven" that given any number that ends with 1, it is either prime or is divisible by 3. / He "proved" the hypothesis.

Figure 1: Interpretation of student's argumentation resorting to the Nyaya approach and to the classical Aristotelian approach

Discussion of example 1.

The student's *personal space* is characterized here by: (1) the experience of the concept of proof in Euclidean geometry; (2) some first explicit information about how a proof is made (thesis, hypothesis, general reasoning, no use of examples); (3) some elements of set-theoretic language in reference to number sets, without deepening of quantification; (4) the interest in showing the own ability (the student was firmly convinced that her/his proof is a good one and s/he wants to prove the truth of the Assertion); (5) the constraint that the text is addressed to the teacher. The teacher's *personal space* is framed at least by the following elements: (I) a valid proof starts from the hypothesis and ends with the thesis; (II) proof is deductive and the use of examples means induction; (III) her spontaneous, implicit, or explicit, use of set-theoretic language as object *language* in mathematical contexts, due

[5] The analysed text was produced with research purposes completely different from the present one; thanks to prof. Paolo Boero from the University of Genoa for having authorized its use for this alternative analysis and for helping to detect the information that was needed to reconstruct the teacher's and the student's personal spaces. We know that the teacher graduated in the mid-1990s at the University of Genoa by a five-years graduation program in Mathematics. In this context, proof is based on classical Aristotelian approach and the object language is always set-theoretic.
[6] The Thesis and the Application of the Nyaya scheme are divided in two parts and the examples are missing.

to her mathematical *forma mentis*. Both *personal spaces* are framed by the assumption that one "uses language in much the same way as everyone else" (Schulz, as quoted by Brown, 1996, p. 121).

The fact that the student has not recognized that the statement is false does not matter; the focus is on her/his reasoning. From the teacher's "classical" point of view, the basis of the student's reasoning could be summed up as follows: The student tries to show that there is a partition of the set of numbers ending with 1 in two subsets: the set A, containing the prime numbers greater than 2 ending with 1, and the set B, containing the multiples of 3 ending with 1. However, s/he does nothing but show that the set of prime numbers ending with 1 is a subset of the set of numbers ending with an odd digit and that there are multiples of 3 ending with 1. Of course, in this way s/he has not proved the existence of the supposed partition, but only the fact that there are two non-empty subsets of the sets A and B, reversing so thesis and hypothesis. Let us now eliminate references to sets in the mathematical sense, that do not belong to everyday reasoning: thinking of the number 3 does not necessarily mean thinking of it as a natural or as a rational number, but as an "object" in itself, in the same way as one thinks of a cup not as an element of the set of all cups, but as an object that falls under the senses.

We see that the student lists some numbers ending with 1, followed by ellipsis, as if this list were to continue. The mathematically shaped thought might interpret this list as the representation of an infinite set. But this list is not necessarily an infinite set in actual sense; it represents probably indeterminacy or vagueness in Peirce's sense or, at most, potential infinity. Indeed, if the student reasons in terms of numerical sets s/he should now try to prove the existence of the supposed partition and s/he does not. But if s/he *does not* reason in terms of numerical sets, what could s/he try to prove? Maybe that given *any* number that ends with 1, *that* number is prime or is a multiple of 3. This reasoning is based on an interpretation of "each" (the universal quantifier) in the sense of "any", that has no meaning in classical FOL but means *generality* in Peirce's sense. The student considers the first numbers listed as *random* cases (any) and finds that they have the required characteristics. This is a not valid generalization both in classical and in Peirce's sense. What the student has shown is that there exist *some* numbers that satisfy the property and so s/he would be able only to quantify recurring to a *vague* existence. This reasoning produces a sort of "fake" generalization by induction. The student knows that the generalization by induction on single cases is not allowed and that s/he must produce a reasoning with general validity (the text is addressed to the teacher). What could *mean* in the student's *personal space* "reasoning that applies in general"? S/he seems simply to produce an *existence* proof, s/he shows that the object being discussed *actually exists* in the sense of the Nyaya logic, and that it is *precise* in Peirce's sense: there are primes (greater than 2) ending with 1 and there are multiples of 3 ending with 1. But the proof is different in the two cases. In the first case s/he shows that the numbers whose existence she wants to prove are a special case of other numbers, "defining" them by next genus (numbers ending with an odd digit) and specific difference (which end with 1). In the second case the proof of existence is made by bringing *examples*. However, s/he does not simply bring examples in the common sense because s/he does not reason on *particular* multiples of 3, but on *some multiples chosen by chance* (they are *vague* in Peirce's sense). To sum up, there seems to be a lack of distinction of vagueness (seen as randomness) and generality (seen as indeterminacy) in Peircean sense. To bridge the gap between every-day-rationality within an empiricist logic (Nyaya) and mathematical rationality, the awareness of this distinction seems to be a necessary condition. Furthermore, the truth concept in the empiricist logic that fits to student's reasoning, seems to be closer to the idea of existence (precise or vague), rather than to the one of generality.

Example 2: Quantification within "blurred" domains.

The second example refers to a classroom argumentation led by the same teacher in another classroom. A worksheet with the argumentation discussed in example 1 is used to show that the proof is not valid. First, the teacher asks to tell if the proof is valid, but the students' attention is captured by the semantical aspects: they detect two counterexamples (121 and 91) and state that it is false. The teacher brings the attention back to validity by asking what the reasoning on the worksheet is.[7]

9	Student 5:	The reasoning is that the multiples of 3 and the prime numbers end with 1.
10	Student 4:	No, that SOME multiple of 3 and SOME prime numbers end with 1. […]
17	Student 8:	Maybe you want to say that … that for CERTAIN prime numbers or multiples of 3 things are going well because they end with 1, but this doesn't mean … […]
19	Student 3:	Yes, the reasoning says only that SOME prime numbers or multiples of 3 end with 1.
20	Student 9:	Even, although if ALL prime numbers or multiples of 3 should end with 1, there could be numbers that end with 1 and ARE NOT prime numbers or multiples of 3.
21	Student 6:	It is as if there is a reversal!
22	Teacher:	S6 said something important: "it is as if there is a reversal". It is an important idea!
23	Student 1:	The hypothesis and the thesis?
24	Student 6:	It seems to me to be of a different matter!
25	Student 4:	To me too, it is a matter … of numbers. Of sets of different numbers. […]
29	Student 9:	I will try to say it again, I don't know if it is OK: the multiples of 3 and the prime numbers are POSSIBLE numbers that end with 1, but these POSSIBLE numbers do not mean that they are ALL the numbers that end with 1.
30	Teacher:	I would say that's it.

Discussion of example 2.

In this example the argumentation is carried out by a group of students. Nevertheless, one can state that the elements (1), (2), (3) and (5) of the student's personal space in example 1 are also elements of the personal spaces of these students because the cultural and formative backgrounds are the same. The element (4) of the student's personal space in the example 1 is substituted by the following one: (4') uncertainty about what validity means in a proof and how it can be accessed, beside by bringing of counter examples. This topic is addressed for the first time in this lesson. The teacher's personal space is the same described in example 1 with the following addition: (IV) intention to focus the discussion on the lack of validity due to a reversal of thesis and hypothesis.[8]

Most of the punctuated words in the transcript are related to quantification but apart from the line 9, the statements show students' struggle with the determination of the domain of validity of the reasoning expressed on the worksheet and of its relation to the domain of the inverse statement which would be a valid one. The non-standard quantification used by the students express the indeterminacy of that domain: SOME, CERTAIN, NOT ALL, POSSIBLE numbers. For instance, as Student 4 (line 10) sums up the reasoning on the worksheet, s/he uses the term *some* as vague existential quantifier in Peirce's sense because s/he knows that there are such numbers (the argumentation on the worksheet tells it) but their multitude is indeterminate; s/he is not able to "close" epistemically a set with this property. In line 22 the teacher supports Student 6's intuition (line 21) that there is a reversal, meaning that the thesis and the hypothesis are reversed, as suggested by Student 1 (line 23). But the students'

[7] In the excerpt we use CAPITAL LETTERS for punctuated words and "…" for pauses longer than 5 seconds.
[8] The points (4') and (IV) are based on a communication made by the researcher that collected the data.

intuition is not a matter of hypothesis and thesis, it is a matter of "numbers", of "sets of different numbers" (lines 24 and 25): There are numbers that satisfy thesis and hypothesis but also numbers that satisfy only the thesis but not the hypothesis. Thus, the inverse statement of the statement to be proved is not a valid inference. This is quite more than what the teacher wanted to put into evidence (reversal of thesis and hypothesis) although it is logically equivalent to it. As in example 1, students' quantification is suitably captured by the Peircean approach that expresses the epistemic uncertainty as vagueness related to variable sets, but unlike in the example 1, the argumentation produces an insight compatible with the teacher's one, related to classical logic. Thus, an investigation about shifts between different logical frames would be useful to better frame the logical analysis.

Conclusions.

According to the hermeneutical approach, the interpretation of the students' behaviour in the examples is meaningful within the global analysis (discussion) and vice versa. Going on in the interpretation, the analysis shows that students spontaneously resort to non-standard logics and non-standard quantification in Peirce's style and that these kinds of quantification and logic allow to formulate an argumentation that explains in a *reliable way* the lack of validity of a proof resorting to blurred domains, not considered within set-theoretic language. In this sense, further research should examine the shifts between different logical frames and the role of the relation between metalanguage and mathematical object-language not only in mathematics (Asenova, 2019), but also in ME. Furthermore, one can state that: (i) The novice's concept of truth might be related to the concept of *existence* of the objects involved in the statement and not to a predicate that it might satisfy: A statement is true if the objects involved in it actually exist; this kind of existence could be "proven" on different levels: by showing one or more "exemplars" with the required characteristics; by referring to single objects as to randomly chosen examples, in a sort of genericity; by referring to a characterisation of the object by a definition by comparison and contrast; (ii) The concept of "reasoning that applies in general" might be related for the student to the *production of a procedure* of a proof of existence, rather than to reasoning that applies to all cases and therefore to no one in particular. All these aspects join some of the students' most recurrent difficulties concerning proof (Stylianides & Stylianides, 2017) and emerged thanks to the non-standard approaches in the analysis.

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